

Homological Mirror Symmetry and Simple Elliptic Singularities

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Abstract

We give a full exceptional collection in the triangulated category of singularities in the sense of Orlov [19] for a hypersurface singularity of Fermat type, and discuss its relation with homological mirror symmetry for simple elliptic hypersurface singularities.

1 Introduction

In [23], Saito defined a simple elliptic singularity to be a normal surface singularity such that the exceptional set of the minimal resolution is a smooth elliptic curve, and classified those which are hypersurface singularities into the following three types:

$$E_6 : x^3 + y^3 + z^3 + \lambda xyz = 0, \quad (1)$$

$$E_7 : x^4 + y^4 + z^2 + \lambda xyz = 0, \quad (2)$$

$$E_8 : x^6 + y^3 + z^2 + \lambda xyz = 0. \quad (3)$$

Here, λ is the parameter which determines the complex structure of the exceptional curve. They form a good class of surface singularities which come next to simple singularities.

Simple singularities are known to be closely related to simple Lie algebras; the Milnor lattice of a simple singularity is isomorphic to the root lattice of the simple Lie algebra of the corresponding type, and the semiuniversal deformation and its simultaneous resolution can be constructed in terms of this Lie algebra after the works of Brieskorn, Grothendieck, and Slodowy (see, e.g., [4]).

To generalize this relation between singularities and Lie algebras to simple elliptic singularities, Saito [24] introduced the notion of an *elliptic root system* by abstracting the properties of vanishing cycles sitting in the Milnor lattice of a simple elliptic singularity. The Lie algebra associated with an

elliptic root system was constructed by Saito and Yoshii [25]. For the elliptic root system coming from the Milnor lattice of a simple elliptic hypersurface singularity, this Lie algebra turns out to be the universal central extension of $\mathfrak{g}[s, s^{-1}, t, t^{-1}]$, where \mathfrak{g} is the simple Lie algebra of the corresponding type. This universal central extension was also studied by Moody, Rao, and Yokonuma [17] under the name of the *2-toroidal algebra*.

Recently, a realization of the “positive part” of the quantized enveloping algebra of an elliptic Lie algebra in terms of the Ringel–Hall algebra of the category of coherent sheaves on a weighted projective line [9] of genus one was found by Schiffmann [26]. The Ringel–Hall algebra of an abelian category is an algebra spanned by isomorphism classes of indecomposable objects as a free abelian group, whose structure constants are given by “counting numbers of extensions.” It was originally used by Ringel [21] to construct the positive part of the quantized enveloping algebra of a simply-laced simple Lie algebra from the abelian category of finite-dimensional representations of a quiver of the corresponding type.

There is also a work of Lin and Peng [16] who used the Ringel–Hall algebra of the *root category* of representations of certain quivers to construct elliptic Lie algebras. Here, the root category is the orbit category of the derived category by twice the shift functor, which was used by Peng and Xiao [20] to answer the problem posed by Ringel [22] of constructing not only the positive part but the whole Lie algebra from the Ringel–Hall algebra. The construction of Lin and Peng is related to that of Schiffmann in that the derived category of representations of their quiver is equivalent to the derived category of coherent sheaves on the weighted projective line used by Schiffmann.

These constructions of elliptic Lie algebras suggest a link between simple elliptic singularities and weighted projective lines of genus one. Elliptic Lie algebras come from elliptic root systems, and in Schiffmann’s work, the elliptic root lattice is realized as the Grothendieck group of the derived category of coherent sheaves on the weighted projective line. In this sense, the derived category of coherent sheaves on a weighted projective line of genus one is a *categorification* of an elliptic root lattice. On the singularity side, the same elliptic root lattice comes as the Milnor lattice of a simple elliptic singularity. This line of thought naturally leads to search for a categorification of the Milnor lattice.

Such a categorification is provided by the *directed Fukaya category* defined by Seidel [27]. It is an A_∞ -category whose set of objects is a distinguished basis of vanishing cycles and whose spaces of morphisms are Lagrangian intersection Floer complexes. Although the directed Fukaya category depends on the choice of a distinguished basis of vanishing cycles, different choices are

related by *mutations*, and the derived category is independent of this choice.

Now a natural question is the relation between the derived category of the directed Fukaya category of a simple elliptic hypersurface singularity and the derived category of coherent sheaves on a weighted projective line of genus one. Since both categories have full exceptional collections, we would like to compare them. The problem then is that there are many possible choices for a full exceptional collection in a triangulated category.

Let us first discuss the singularity side. Fix a field k . For a positive integer p greater than one, let $W_p(X) \in \mathbb{C}[X]$ be a general polynomial of degree p in one variable. Then for a suitable choice of a distinguished basis of vanishing cycles, the corresponding directed Fukaya category $\mathfrak{Fuk}^{\rightarrow} W_p$ over k has $p - 1$ objects $(L_i)_{i=1}^{p-1}$, and the spaces of morphisms are

$$\mathrm{hom}_{\mathfrak{Fuk}^{\rightarrow} W_p}(L_i, L_j) = \begin{cases} k \cdot \mathrm{id}_{L_i} & j = i, \\ k[-1] & j = i + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $k[-1]$ is the one-dimensional graded vector space concentrated in degree 1 [28, section (2B)]. Note that $\mathfrak{Fuk}^{\rightarrow} W_p$ is not only an A_{∞} -category but also a DG -category with a trivial differential, since all the A_{∞} -operations are necessarily trivial.

Now for positive integers p_0, \dots, p_n greater than one, consider the polynomial

$$W_{p_0, \dots, p_n}(X_0, \dots, X_n) = W_{p_0}(X_0) + \dots + W_{p_n}(X_n) \quad (4)$$

in $n + 1$ variables, which defines a holomorphic map from \mathbb{C}^{n+1} to \mathbb{C} . By equipping \mathbb{C}^{n+1} with the standard symplectic structure, W gives an exact Lefschetz fibration in the sense of Seidel [29], and one can consider its directed Fukaya category $\mathfrak{Fuk}^{\rightarrow} W_{p_0, \dots, p_n}$. The following conjecture is a special case of [1, Conjecture 1.3]:

Conjecture 1. *There exists an equivalence*

$$D^b \mathfrak{Fuk}^{\rightarrow} W_{p_0, \dots, p_n} \cong D^b(\mathfrak{Fuk}^{\rightarrow} W_{p_0} \otimes \dots \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_n}) \quad (5)$$

of triangulated categories.

This conjecture is known to hold when $n = 1$ [1, section 6.3], and the general case is under investigation by Auroux, Katzarkov, Orlov, and Seidel. With this in mind, we consider $D^b(\mathfrak{Fuk}^{\rightarrow} W_{p_0} \otimes \dots \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_n})$ instead of $D^b \mathfrak{Fuk}^{\rightarrow} W_{p_0, \dots, p_n}$ in this paper, which clearly has a full exceptional collection $(L_1 \otimes \dots \otimes L_1, \dots, L_{p_0-1} \otimes \dots \otimes L_{p_n-1})$.

Let us next discuss the weighted projective side. For a sequence $\mathbf{p} = (p_0, p_1, p_2)$ of integers greater than one, put

$$A(\mathbf{p}) = k[x, y, z]/(x^{p_0} + y^{p_1} + z^{p_2}). \quad (6)$$

Let $L(\mathbf{p})$ be the abelian group generated by four elements $\vec{x}, \vec{y}, \vec{z}$, and \vec{c} with the relation $p_0\vec{x} = p_1\vec{y} = p_2\vec{z} = \vec{c}$, and equip $A(\mathbf{p})$ with an $L(\mathbf{p})$ -grading by setting $\deg x = \vec{x}$, $\deg y = \vec{y}$, and $\deg z = \vec{z}$.

Definition 2. The category $\text{qgr}A(\mathbf{p})$ of coherent sheaves on the weighted projective line of weight $\mathbf{p} = (p_0, p_1, p_2)$ is the quotient category

$$\text{qgr}A(\mathbf{p}) = \text{gr-}A(\mathbf{p})/\text{tor-}A(\mathbf{p})$$

of the category $\text{gr-}A(\mathbf{p})$ of finitely-generated $L(\mathbf{p})$ -graded $A(\mathbf{p})$ -modules by its full subcategory $\text{tor-}A(\mathbf{p})$ consisting of torsion modules.

This definition is equivalent to that of Geigle and Lenzing by the Serre's Theorem in [9, section 1.8].

The first candidate of a full exceptional collection on a weighted projective line is the Beilinson-type generator provided by [9, Proposition 4.1]. However, this collection does not match nicely with the collection $(L_1 \otimes L_1 \otimes L_1, \dots, L_{p_0-1} \otimes L_{p_1-1} \otimes L_{p_2-1})$ on the singularity side, and one needs another method to find a full exceptional collection on the weighted projective line.

Such a method is provided by a theorem of Orlov [19] relating the derived category of coherent sheaves to the *triangulated category of singularities*. The origin of this category goes back to the work of Eisenbud on matrix factorizations [7], which was revived by Kapustin and Li [12] and Orlov [18] motivated by an idea of Kontsevich. The concept of grading was introduced by Hori and Walcher [10], and the relation with the derived category of coherent sheaves was conjectured by Walcher [32, section 4.7]. The idea that triangulated categories of singularities may be useful in constructing Lie algebras from hypersurface singularities is due to Takahashi [30] and Kajiwara, Saito, and Takahashi [11]. Although the above works deal only with gradings by \mathbb{Z} , it is straightforward to extend the construction to gradings by slightly more involved groups which are extensions of finite abelian groups by \mathbb{Z} , and these extra structures provide a link with weighted projective lines.

Definition 3. The triangulated category of singularities $D_{\text{Sg}}^{\text{gr}}(A(\mathbf{p}))$ is the quotient category

$$D_{\text{Sg}}^{\text{gr}}(A(\mathbf{p})) = D^b(\text{gr-}A(\mathbf{p}))/D^b(\text{grproj-}A(\mathbf{p})).$$

of the bounded derived category $D^b(\text{gr-}A(\mathbf{p}))$ of finitely-generated $L(\mathbf{p})$ -graded $A(\mathbf{p})$ -modules by its full triangulated subcategory $D^b(\text{grproj-}A(\mathbf{p}))$ consisting of perfect complexes, i.e., bounded complexes of projective modules.

The following theorem is obtained by a straightforward adaptation of Orlov's argument in [19] to the $L(\mathbf{p})$ -graded situation:

Theorem 4. *For $\mathbf{p} = (3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$, there exists an equivalence*

$$D_{\text{Sg}}^{\text{gr}}(A(\mathbf{p})) \cong D^b(\text{qgr}A(\mathbf{p}))$$

of triangulated categories.

The advantage of working with $D_{\text{Sg}}^{\text{gr}}(A(\mathbf{p}))$ instead of $D^b(\text{qgr}A(\mathbf{p}))$ is that one can easily find an exceptional collection $(E_1, \dots, E_{(p_0-1)(p_1-1)(p_2-1)})$ whose total morphism algebra $\bigoplus_{i,j=1}^{(p_0-1)(p_1-1)(p_2-1)} \text{Ext}^*(E_i, E_j)$ is isomorphic to that of $(L_1 \otimes L_1 \otimes L_1, \dots, L_{p_0-1} \otimes L_{p_1-1} \otimes L_{p_2-1})$. One can also show that this exceptional collection is *full*, i.e., generates the whole triangulated category, and that the *DG*-algebra underlying this graded algebra is *formal*, i.e., quasi-isomorphic to its cohomology. These facts prove the following:

Theorem 5. *For a sequence $\mathbf{p} = (p_0, p_1, p_2)$ of integers greater than one, there exists an equivalence*

$$D_{\text{Sg}}^{\text{gr}}(A(\mathbf{p})) \cong D^b(\mathfrak{Fuk}^{\rightarrow} W_{p_0} \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_1} \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_2})$$

of triangulated categories.

By combining Theorem 4 with Theorem 5, one obtains

$$D^b(\text{qgr}A(\mathbf{p})) \cong D^b(\mathfrak{Fuk}^{\rightarrow} W_{p_0} \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_1} \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_2}) \quad (7)$$

for $(p_0, p_1, p_2) = (3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$. The right-hand side of (7) is expected, by Conjecture 1, to be equivalent to the derived category $D^b \mathfrak{Fuk}^{\rightarrow} W_{p_0, p_1, p_2}$ of the directed Fukaya category of W_{p_0, p_1, p_2} , which is a deformation of a simple elliptic singularity. As a slogan, *the mirrors of the weighted projective lines of weights $(3, 3, 3)$, $(2, 4, 4)$, and $(2, 3, 6)$ are simple elliptic hypersurface singularities*. When $\mathbf{p} = (2, 4, 4)$ or $(2, 3, 6)$, one can state as a theorem that the mirror of the weighted projective line of weight \mathbf{p} is the exact Lefschetz fibration given by W_{p_1, p_2} , since Conjecture 1 is known to hold for $n = 1$ and tensoring with $\mathfrak{Fuk}^{\rightarrow} W_2$ does not change the category. This adds two more examples to the list [1, 2, 28, 31] of spaces with a full exceptional collection

where the *homological mirror symmetry* conjecture of Kontsevich [14, 15] is known to hold.

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2 Generalities on triangulated categories

In this section, we recall the definitions of admissible subcategories, semiorthogonal decompositions, exceptional collections, and enhanced triangulated categories following Bondal and Kapranov [3] and Orlov [19]. Throughout this section, we assume that all categories are small.

Definition 6. Let \mathcal{N} be a full triangulated subcategory of a triangulated category \mathcal{T} . \mathcal{N} is right (resp. left) admissible if the inclusion functor $i : \mathcal{N} \hookrightarrow \mathcal{T}$ has a right (resp. left) adjoint functor. \mathcal{N} is admissible if it is right and left admissible.

The importance of the concept of admissibility lies in the following fact:

Lemma 7. Let \mathcal{N} be a full triangulated subcategory in a triangulated category \mathcal{T} . If \mathcal{N} is right (resp. left) admissible, then the quotient category \mathcal{T}/\mathcal{N} is equivalent to \mathcal{N}^\perp (resp. ${}^\perp\mathcal{N}$).

Here, \mathcal{N}^\perp (resp. ${}^\perp\mathcal{N}$) denotes the right (resp. left) orthogonal to \mathcal{N} , i.e., the full subcategory consisting of all objects M such that $\text{Hom}(N, M) = 0$ (resp. $\text{Hom}(M, N) = 0$) for all $N \in \mathcal{N}$.

Admissible subcategories give *semiorthogonal decompositions* of triangulated categories:

Definition 8. A sequence of full triangulated subcategories $(\mathcal{N}_0, \dots, \mathcal{N}_n)$ in a triangulated category \mathcal{T} is a weak semiorthogonal decomposition if there exists a sequence of left admissible subcategories $\mathcal{N}_0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \dots \subset \mathcal{T}_n = \mathcal{T}$ such that \mathcal{N}_p is left orthogonal to \mathcal{T}_{p-1} in \mathcal{T}_p . We write this as $\mathcal{T} = \langle \mathcal{N}_0, \dots, \mathcal{N}_n \rangle$.

A *full exceptional collection* is a generator of a triangulated category with good properties:

Definition 9. Let \mathcal{T} be a triangulated category over a field k .

1. An object E of \mathcal{T} is exceptional if

$$\mathrm{Ext}^i(E, E) = \begin{cases} k & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. An ordered set of objects (E_0, \dots, E_n) is an exceptional collection if all the E_i 's are exceptional and

$$\mathrm{Ext}^l(E_i, E_j) = 0$$

for all $i > j$ and $l \in \mathbb{Z}$.

3. An exceptional collection (E_0, \dots, E_n) is full if they generate \mathcal{T} as a triangulated category.

Given a full exceptional collection (E_0, \dots, E_n) , one might hope to reconstruct \mathcal{T} from its total morphism algebra

$$\bigoplus_{i,j=0}^n \mathrm{Ext}^*(E_i, E_j).$$

Unfortunately, this in general is not possible due to the loss of information coming from the triangulated structure of \mathcal{T} , such as Massey products. To remedy this situation, Bondal and Kapranov [3] introduced the concept of an *enhancement* of a triangulated category. The philosophy of derived categories is to deal with complexes rather than their cohomologies, to keep track of the information which is lost in passing from complexes to their cohomologies. To push this philosophy further, one has to treat not only objects but also the spaces of morphisms as complexes:

Definition 10. Let k be a field. A *differential graded category* over k , or a *DG-category*, is a category such that

1. the set $\mathrm{hom}(E, F)$ of morphisms between two objects E and F is a complex of k -vector space, i.e., has a \mathbb{Z} -grading $\mathrm{hom}(E, F) = \bigoplus_{k \in \mathbb{Z}} \mathrm{hom}^k(E, F)$ and a differential d of degree 1,
2. for any object E , the identity morphism $\mathrm{id}_E \in \mathrm{hom}(E, E)$ satisfies $d(\mathrm{id}_E) = 0$, and
3. the composition

$$\circ : \mathrm{hom}(E, F) \times \mathrm{hom}(F, G) \rightarrow \mathrm{hom}(E, G)$$

is a k -bilinear map preserving the grading, and satisfies the Leibniz rule

$$d(a \circ b) = d(a) \circ b + (-1)^{\deg a} a \circ d(b).$$

A *DG-functor* between two *DG*-categories is a k -linear functor which preserves the gradings and commutes with the differentials.

For a nice review on *DG*-categories, see e.g., [13]. With a *DG*-category \mathcal{D} , one can associate its cohomology category $H^0(\mathcal{D})$ whose set of objects is the same as \mathcal{D} and whose set $\text{Hom}(E, F)$ of morphisms is the zeroth cohomology group of $\text{hom}(E, F)$.

Given two *DG*-categories, one can consider its tensor product:

Definition 11. The tensor product $\mathcal{D}_1 \otimes \mathcal{D}_2$ of two *DG*-categories \mathcal{D}_1 and \mathcal{D}_2 has the set of objects

$$\mathfrak{Ob}(\mathcal{D}_1) \times \mathfrak{Ob}(\mathcal{D}_2)$$

and the space of morphisms

$$\text{hom}_{\mathcal{D}_1 \otimes \mathcal{D}_2}((E_1, E_2), (F_1, F_2)) = \text{hom}_{\mathcal{D}_1}(E_1, F_1) \otimes \text{hom}_{\mathcal{D}_2}(E_2, F_2)$$

with the composition

$$(f \otimes v) \circ (g \otimes w) = (-1)^{\deg v \cdot \deg g} (f \circ g) \otimes (v \circ w)$$

and the differential

$$d(f \otimes v) = (df) \otimes v + (-1)^{\deg f} f \otimes (dv).$$

The following definition is due to Bondal and Kapranov:

Definition 12 ([3, §1. Definition 1]). A *twisted complex* over a *DG*-category \mathcal{D} is a set $\{\{E_i\}_{i \in \mathbb{Z}}, \{q_{ij}\}_{i,j \in \mathbb{Z}}\}$, where E_i 's are objects of \mathcal{D} equal to 0 for almost all i , and q_{ij} is an element of $\text{hom}^{i-j+1}(E_i, E_j)$ satisfying $dq_{ij} + \sum_k q_{kj} \circ q_{ik} = 0$. A twisted complex $\{E_i, q_{ij}\}$ is called *one-sided* if $q_{ij} = 0$ for $i \geq j$.

We will assume that all twisted complexes are one-sided henceforth. Twisted complexes over a *DG*-category \mathcal{D} form a *DG*-category by

$$\text{hom}^k(\{E_i, q_{ij}\}, \{F_i, r_{ij}\}) = \bigoplus_{l+i-i=k} \text{hom}_{\mathcal{D}}^l(E_i, F_j)$$

and

$$df = d_{\mathcal{D}}f + \sum_m (r_{jm} \circ f + (-1)^{l(i-m+1)} f \circ q_{mi})$$

for $f \in \text{hom}_{\mathcal{D}}^l(E_i, F_j)$. Let \mathcal{D}^{\oplus} be the *DG*-category obtained from \mathcal{D} by formally adjoining finite direct sums, and $\text{Pre-Tr}(\mathcal{D})$ be the *DG*-category

of twisted complexes over \mathcal{D}^\oplus . The cohomology category $H^0(\text{Pre-Tr}(\mathcal{D}))$ will be denoted by $D^b(\mathcal{D})$. A twisted complex $K = \{E_i, q_{ij}\} \in \text{Pre-Tr}(\mathcal{D})$ defines a contravariant DG -functor from \mathcal{D} to the DG -category of complexes of k -vector spaces by sending $E \in \mathfrak{Ob}(\mathcal{D})$ to $\text{hom}_{\text{Pre-Tr}(\mathcal{D})}(E, K)$. Here, E is considered as a twisted complex $\{E, 0\}$ concentrated at degree zero.

Definition 13 ([3, §3. Definition 1]). A DG -category \mathcal{D} is *pretriangulated* if for every twisted complex $K \in \mathfrak{Ob}(\text{Pre-Tr}(\mathcal{D}))$, the corresponding contravariant DG -functor is representable.

By [3, §3. Proposition 2], the cohomology category $H^0(\mathcal{D})$ of a pretriangulated DG -category \mathcal{D} has a natural structure of a triangulated category.

Definition 14. An *enhancement* of a triangulated category \mathcal{T} is a pretriangulated DG -category \mathcal{D} together with an equivalence $H^0(\mathcal{D}) \rightarrow \mathcal{T}$ of triangulated categories.

A typical example of an enhancement is the DG -category underlying the bounded derived category of an abelian category with enough injectives or projectives. [3, §3. Example 3].

Now we can state the reconstruction of a triangulated category from a finite number of generators:

Theorem 15 (Bondal–Kapranov [3, §4. Theorem 1]). *Assume that a triangulated category \mathcal{T} generated by a finite number of objects E_0, \dots, E_n has an enhancement \mathcal{D} . Let \mathcal{A} be the full DG -subcategory of \mathcal{D} consisting of E_0, \dots, E_n . Then one has an equivalence*

$$\mathcal{T} \cong D^b(\mathcal{A})$$

of triangulated categories.

3 Triangulated categories of singularities

We prove Theorem 4 in this section. The proof is a straightforward adaptation of arguments from [19] to the $L(\mathbf{p})$ -graded situation, which we include for the reader's convenience. For an $L(\mathbf{p})$ -graded $A(\mathbf{p})$ -module M and $\vec{n} \in L(\mathbf{p})$, $M(\vec{n})$ will denote the graded $A(\mathbf{p})$ -module obtained by shifting the grading by \vec{n} ; $M(\vec{n})_{\vec{m}} = M_{\vec{n}+\vec{m}}$. Put $\mathbf{p} = (p_0, p_1, p_2) = (3, 3, 3), (2, 4, 4)$, or $(2, 3, 6)$, $A = A(\mathbf{p})$, and $L = L(\mathbf{p})$. Define $\phi : L \rightarrow \mathbb{Z}$ by

$$\phi(\vec{x}) = \frac{p_2}{p_0}, \quad \phi(\vec{y}) = \frac{p_2}{p_1}, \quad \phi(\vec{z}) = 1,$$

and let $\mathcal{S}_{<0}$ and $\mathcal{S}_{\geq 0}$ be the full triangulated subcategories of $D^b(\text{gr-}A)$ generated by $k(\vec{n})$ for $\phi(\vec{n}) > 0$ and $\phi(\vec{n}) \leq 0$ respectively. Here k denotes the A -module $A/(x, y, z)$. Let further $\mathcal{P}_{<0}$ and $\mathcal{P}_{\geq 0}$ be the full triangulated subcategories of $D^b(\text{gr-}A)$ generated by the modules $A(\vec{n})$ for $\phi(\vec{n}) > 0$ and $\phi(\vec{n}) \leq 0$ respectively. Then one has weak semiorthogonal decompositions

$$D^b(\text{gr-}A) = \langle \mathcal{S}_{<0}, D^b(\text{gr-}A_{\geq 0}) \rangle, \quad (8)$$

$$D^b(\text{gr-}A) = \langle D^b(\text{gr-}A_{\geq 0}), \mathcal{P}_{<0} \rangle, \quad (9)$$

where $\text{gr-}A_{\geq 0}$ is the full subcategory of $\text{gr-}A$ consisting of $L(\mathbf{p})$ -graded A -modules M such that $M_{\vec{n}} = 0$ if $\phi(\vec{n}) < 0$. The corresponding statement in the \mathbb{Z} -graded case is [19, Lemma 2.3], whose proof also works here verbatim. This shows that

$${}^\perp \mathcal{S}_{<0} = \mathcal{P}_{<0}^\perp.$$

Since A is Gorenstein, the duality functor

$$D = \mathbb{R} \text{Hom}_A(-, A) : D^b(\text{gr-}A)^\circ \rightarrow D^b(\text{gr-}A)$$

is an equivalence and maps $(\mathcal{P}_{\leq 0})^\circ$ to $\mathcal{P}_{\geq 0}$. Moreover, since

$$\mathbb{R} \text{Hom}_A(k, A) = k(\vec{x} + \vec{y} + \vec{z} - \vec{c})[-2]$$

and

$$\phi(\vec{x} + \vec{y} + \vec{z} - \vec{c}) = 0$$

in our case, D also gives an equivalence between $(\mathcal{S}_{\leq 0})^\circ$ and $\mathcal{S}_{\geq 0}$. Therefore, $\mathcal{S}_{\geq 0}$ is right admissible in $D^b(\text{gr-}A)$, $\mathcal{P}_{\geq 0}$ is left admissible in $D^b(\text{gr-}A)$, and

$$\mathcal{S}_{\geq 0}^\perp = {}^\perp \mathcal{P}_{\geq 0}.$$

Moreover, $\mathcal{S}_{<0}$ and $\mathcal{P}_{\geq 0}$ are mutually orthogonal. Hence one has weak semiorthogonal decompositions

$$D^b(\text{gr-}A) = \langle \mathcal{S}_{<0}, \mathcal{D}_0, \mathcal{S}_{\geq 0} \rangle$$

and

$$D^b(\text{gr-}A) = \langle \mathcal{S}_{<0}, \mathcal{P}_{\geq 0}, \mathcal{T}_0 \rangle,$$

such that $\mathcal{D}_0 \cong D^b(\text{qgr} A)$ and $\mathcal{T}_0 \cong D_{\text{Sg}}^{\text{gr}}(A)$. Now one has

$$\begin{aligned} D^b(\text{gr-}A) &= \langle \mathcal{S}_{<0}, \mathcal{D}_0, \mathcal{S}_{\geq 0} \rangle \\ &= \langle \mathcal{P}_{\geq 0}, \mathcal{S}_{<0}, \mathcal{D}_0 \rangle \\ &= \langle \mathcal{S}_{<0}, \mathcal{P}_{\geq 0}, \mathcal{D}_0 \rangle \\ &= \langle \mathcal{S}_{<0}, \mathcal{P}_{\geq 0}, \mathcal{T}_0 \rangle, \end{aligned}$$

which shows $\mathcal{T}_0 = \mathcal{D}_0$ and hence

$$D_{\text{Sg}}^{\text{gr}}(A) \cong D^b(\text{qgr} A).$$

□

4 A full exceptional collection

We prove Theorem 5 in this section. Fix any weight $\mathbf{p} = (p_0, p_1, p_2)$ and put $A = A(\mathbf{p})$ and $L = L(\mathbf{p})$. We will find a full triangulated subcategory \mathcal{T} of $D^b(\text{gr-}A)$ equivalent to $D_{\text{Sg}}^{\text{gr}}(A)$ such that $(k(\vec{n}))_{\vec{n} \in I}$ is a full exceptional collection in \mathcal{T} , where $k = A/(x, y, z)$ and

$$I = \{a\vec{x} + b\vec{y} + c\vec{z} \in L \mid -p_0 + 2 \leq a \leq 0, -p_1 + 2 \leq b \leq 0, -p_2 + 2 \leq c \leq 0\}.$$

Let L_+ be the subset of L defined by

$$L_+ = \{-2\vec{c} + a\vec{x} + b\vec{y} + c\vec{z} \mid a \geq 1, b \geq 1, c \geq 1\},$$

and L_- be the complement $L \setminus L_+$. Let further \mathcal{S}_- and \mathcal{P}_+ be the full triangulated subcategories of $D^b(\text{gr-}A)$ generated by $k(\vec{n})$ for $\vec{n} \in L_+$ and $A(\vec{m})$ for $\vec{m} \in L_-$ respectively. Then \mathcal{S}_- and \mathcal{P}_+ are left admissible in $D^b(\text{gr-}A)$, and since $\mathcal{P}_+ \subset {}^\perp \mathcal{S}_-$, one has a weak semiorthogonal decomposition

$$D^b(\text{gr-}A) = \langle \mathcal{S}_-, \mathcal{P}_+, \mathcal{T} \rangle$$

such that $\mathcal{T} \cong D_{\text{Sg}}^{\text{gr}}(A)$. One can see that $k(\vec{n})$ for $\vec{n} \in I$ belongs to \mathcal{T} , since

$$\mathbb{R} \text{Hom}(k(\vec{m}), k(\vec{n})) = 0$$

if $\vec{m} \notin \vec{n} + \mathbb{N}\vec{x} + \mathbb{N}\vec{y} + \mathbb{N}\vec{z}$, and

$$\mathbb{R} \text{Hom}(k(\vec{m}), A(\vec{n})) = 0$$

if $\vec{m} \neq -\vec{c} + \vec{x} + \vec{y} + \vec{z} + \vec{n}$. The $\mathbb{R} \text{Hom}$'s between them can be calculated by the following free resolution

$$\begin{array}{ccccc}
 & A(-\vec{c} - \vec{y} - \vec{z}) & & A(-\vec{x} - \vec{y} - \vec{z}) & & A(-\vec{y} - \vec{z}) \\
 & \oplus & & \oplus & & \oplus \\
 & A(-\vec{c} - \vec{x} - \vec{z}) & & A(-\vec{c} - \vec{z}) & & A(-\vec{x} - \vec{z}) \\
 \dots \longrightarrow & \oplus & \xrightarrow{d_4} & \oplus & \xrightarrow{d_3} & \oplus \\
 & A(-\vec{c} - \vec{x} - \vec{y}) & & A(-\vec{c} - \vec{y}) & & A(-\vec{x} - \vec{y}) \\
 & \oplus & & \oplus & & \oplus \\
 & A(-2\vec{c}) & & A(-\vec{c} - \vec{x}) & & A(-\vec{c}) \\
 & & & A(-\vec{z}) & & \\
 & & & \oplus & & \\
 & & & \xrightarrow{d_2} A(-\vec{y}) & \xrightarrow{d_1} A \longrightarrow k, & \\
 & & & \oplus & & \\
 & & & A(-\vec{x}) & &
 \end{array}$$

where

$$d_1 = \begin{pmatrix} z & y & x \end{pmatrix}, \quad (10)$$

$$d_2 = \begin{pmatrix} -y & -x & 0 & z^{p_2-1} \\ z & 0 & -x & y^{p_1-1} \\ 0 & z & y & x^{p_0-1} \end{pmatrix}, \quad (11)$$

$$d_3 = \begin{pmatrix} x & -y^{p_1-1} & z^{p_2-1} & 0 \\ -y & -x^{p_0-1} & 0 & z^{p_2-1} \\ z & 0 & -x^{p_0-1} & y^{p_1-1} \\ 0 & z & y & x \end{pmatrix}, \quad (12)$$

$$d_4 = \begin{pmatrix} x^{p_0-1} & -y^{p_1-1} & z^{p_2-1} & 0 \\ -y & -x & 0 & z^{p_2-1} \\ z & 0 & -x & y^{p_1-1} \\ 0 & z & y & x^{p_0-1} \end{pmatrix}, \quad (13)$$

to be

$$\begin{aligned} & \mathbb{R} \operatorname{Hom}(k(a_1\vec{x} + b_1\vec{y} + c_1\vec{z}), k(a_2\vec{x} + b_2\vec{y} + c_2\vec{z})) \\ &= \begin{cases} k & (a_2, b_2, c_2) = (a_1, b_1, c_1), \\ k[-1] & (a_2, b_2, c_2) = (a_1 - 1, b_1, c_1), (a_1, b_1 - 1, c_1), (a_1, b_1, c_1 - 1), \\ k[-2] & (a_2, b_2, c_2) = (a_1 - 1, b_1 - 1, c_1), (a_1 - 1, b_1, c_1 - 1), (a_1, b_1 - 1, c_1 - 1), \\ k[-3] & (a_2, b_2, c_2) = (a_1 - 1, b_1 - 1, c_1 - 1), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here, $k[i]$ for $i \in \mathbb{Z}$ denotes the one-dimensional vector space concentrated in degree $-i$. Therefore $(k(\vec{n}))_{\vec{n} \in I}$ is an exceptional collection. It is straightforward to read off the structure of the Yoneda products from the above resolution to show that the full subcategory of \mathcal{T} consisting of $(k(\vec{n}))_{\vec{n} \in I}$ is isomorphic as a graded category to $\mathfrak{Fuk}^{\rightarrow} W_{p_0} \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_1} \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_2}$. Moreover, \mathcal{T} has an enhancement induced from that of $D^b(\operatorname{gr} A)$, and one can see using the above resolution again that the total morphism DG -algebra $\bigoplus_{\vec{m}, \vec{n} \in I} \operatorname{hom}(k(\vec{m}), k(\vec{n}))$ of the above exceptional collection is *formal*, i.e., quasi-isomorphic to its cohomology with the trivial differential as a DG -algebra. This shows that $D^b(\mathfrak{Fuk}^{\rightarrow} W_{p_0} \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_1} \otimes \mathfrak{Fuk}^{\rightarrow} W_{p_2})$ is equivalent to the full triangulated subcategory of $D_{\operatorname{Sg}}^{\operatorname{gr}}(A(\mathbf{p}))$ generated by the above exceptional collection.

Now we prove that the image of $(k(\vec{n}))_{\vec{n} \in I}$ in $D_{\operatorname{Sg}}^{\operatorname{gr}}(A)$ is full. Let $G_0 = \operatorname{Spec} k[L/\mathbb{Z}\vec{c}]$ be the subgroup of $G = \operatorname{Spec} k[L]$ isomorphic to $(\mathbb{Z}/p_0\mathbb{Z}) \times (\mathbb{Z}/p_1\mathbb{Z}) \times (\mathbb{Z}/p_2\mathbb{Z})$. Since L is the group of characters of G , the L -grading of A defines an action of G on A , and an L -graded A -module is the same

thing as a G -equivariant A -module. For a G_0 -module M , let M^{G_0} denote its G_0 -invariant part. The restriction of the action of G on $\text{Spec } A$ to G_0 is given by

$$\begin{array}{ccc} G_0 \ni ([i], [j], [k]) : & \text{Spec } A & \longrightarrow \text{Spec } A \\ & \downarrow & \downarrow \\ & (x, y, z) & \longmapsto (\zeta_{p_0}^i x, \zeta_{p_1}^j y, \zeta_{p_2}^k z), \end{array}$$

where $\zeta_n = \exp[2\pi\sqrt{-1}/n]$ for a positive integer n . Let $X \subset \text{Spec } A$ be the closed subscheme defined by the ideal (xyz) where the action of G is not free.

By [18, Lemma 1.11], an element of $D_{\text{Sg}}^{\text{gr}}(A)$ is isomorphic to the shift $M[i]$ of a finitely-generated L -graded A -module M by some integer i . Let M_{xyz} be the localization of M by $xyz \in A$. Then $M_{xyz}^{G_0}$ is an $A_{xyz}^{G_0}$ -module and one has $M_{xyz}^{G_0} \otimes_{A_{xyz}^{G_0}} A_{xyz} \cong M_{xyz}$, since the action of G_0 on $\text{Spec } A_{xyz}$ is free. Take a finitely-generated A^{G_0} -module N such that $N_{x^{p_0}y^{p_1}z^{p_2}} \cong (M_{xyz})^{G_0}$. Then one has an isomorphism $(N \otimes_{A^{G_0}} A)_{xyz} \rightarrow M_{xyz}$, which can be extended to a morphism $N \otimes_{A^{G_0}} A \rightarrow M$ by multiplying a power of xyz if necessary. Since $A^{G_0} = k[x^{p_0}, y^{p_1}, z^{p_2}]/(x^{p_0} + y^{p_1} + z^{p_2})$ is regular and A is flat over A^{G_0} , $N \otimes_{A^{G_0}} A$ is perfect. Therefore, by replacing M with the kernel and the cokernel of the above morphism, one can assume that M is supported on X .

Since X is the union of G -invariant subschemes X_0 , X_1 , and X_2 defined by $x = 0$, $y = 0$, and $z = 0$ respectively, one can assume that the support of M is contained in the subscheme $X_0 \cong \text{Spec } A/(x)$. (In general, a coherent sheaf supported on the union $S_1 \cup S_2$ of closed subschemes can be obtained from coherent sheaves supported on S_1 and S_2 by taking cones.) For a finitely-generated A -module M supported on X_0 , let $l(M)$ be the minimal integer n such that $x^n M = 0$. Then one has an exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M/xM \rightarrow 0$$

such that M/xM is an $A/(x)$ -module and $l(M') < l(M)$. By replacing M with M' and continuing this process, one can assume that M is an $A/(x)$ -module.

Since x is not a zero divisor of A , $A/(x)$ is perfect by the following free resolution:

$$0 \rightarrow A \rightarrow A \rightarrow A/(x) \rightarrow 0.$$

Therefore, a perfect $A/(x)$ -module is also perfect as an A -module. By replacing A with $A/(x) = k[y, z]/(y^{p_1} + z^{p_2})$ and repeating the same argument, one can see that any L -graded $A/(x)$ -module can be obtained from $A/(x, y)$ -modules by taking cones up to perfect complexes. Since the same is true for $A/(y)$ -modules and $A/(z)$ -modules, $D_{\text{Sg}}^{\text{gr}}(A)$ is generated by modules supported at $x = y = z = 0$ as a triangulated category.

Since any module supported at $x = y = z = 0$ can be obtained from $k = A/(x, y, z)$ by extensions, it is enough to show that all sheaves of the form $k(\vec{n})$ for $\vec{n} \in L$ can be obtained from $(k(\vec{n}))_{\vec{n} \in I}$ by taking cones up to perfect complexes. To do this, first note that the exact sequences

$$\begin{aligned} 0 \rightarrow k(-\vec{z}) &\rightarrow k[z]/(z^2) \rightarrow k \rightarrow 0, \\ 0 \rightarrow k(-2\vec{z}) &\rightarrow k[z]/(z^3) \rightarrow k[z]/(z^2) \rightarrow 0, \\ &\vdots \\ 0 \rightarrow k(-(p_2 - 1)\vec{z}) &\rightarrow k[z]/(z^{p_2}) \rightarrow k[z]/(z^{p_2-1}) \rightarrow 0 \end{aligned}$$

of A -modules show that $k(-(p_2 - 1)\vec{z})$ can be obtained from $k, k(-\vec{z}), \dots, k(-(p_2 - 2)\vec{z})$ by taking cones up to the perfect module $k[z]/(z^{p_2}) \cong A/(x, y)$. Then by shifting the degrees, one can see that for any $\vec{n} \in L$, $k(\vec{n})$ can be obtained from either $k(\vec{n} - \vec{z}), k(\vec{n} - 2\vec{z}), \dots, k(\vec{n} - (p_2 - 1)\vec{z})$ or $k(\vec{n} + \vec{z}), k(\vec{n} + 2\vec{z}), \dots, k(\vec{n} + (p_2 - 1)\vec{z})$ by taking cones up to perfect complexes. The same is true for \vec{x} and \vec{y} , which proves that $(k(\vec{n}))_{\vec{n} \in I}$ is full.

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